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Note

On a problem of Erdős and Graham

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Abstract

In this paper we shall answer a question of Erdős and Graham (1980, p. 18) concerning sums of integer sequences. Furthermore, we shall examine for what sequences $(r_i, c_i)_{i=1}^{\infty}$ it is true that if $B = (b_i)$ is a sequence of natural numbers such $b_{i+1} \geqslant r_i b_i - c_i$ then, for some sequence $A = (a_i)_{i=1}^{\infty}$ of natural numbers with $2 \leqslant a_{i+1} - a_i \leqslant 3$, we have $(A + A) \cap B \neq \emptyset$.

1. Introduction

In 1980, Erdős and Graham [2, p. 18] noticed that if B is a fairly fast lacunary sequence of integers, then for some dense set $A \subset \mathbb{N}$ we have $(A + A) \cap B = \emptyset$. To be precise, they remarked that if $B = (b_i)_1^{\infty}$ with $1 \leqslant b_1 < b_2 < \cdots$ and $b_{i+1} \geqslant 2b_i$, then for some infinite sequence $A = (a_i)_1^{\infty}$, with $1 \leqslant a_1 < a_2 < \cdots$ and $2 \leqslant a_{i+1} - a_i \leqslant 3$, we have $(A + A) \cap B = \emptyset$. Here and in the rest of this paper, all sequences are strictly increasing sequences of natural numbers, unless mentioned otherwise. As usual, for a sequence $S = (s_i)_{i=1}^{\infty}$ we write $S + S = \{s_i + s_j : \ 1 \leqslant i \leqslant j < \infty\}$.

In connection with the above observation, Erdős and Graham [2] also posed the question whether the sequence A can be chosen to satisfy $(A + A + A) \cap B = \emptyset$. Our aim in this paper is to show that such an A need not exist. In fact, we shall show that such an A need not exist even when B has considerably greater lacunarity.

We shall also examine how far the condition $b_{i+1} \geqslant 2b_i$ can be relaxed to allow us to conclude the existence of an appropriate sequence A. We shall show that if we demand that $b_{i+1} \geqslant 2b_i - c_i$, with $0 \leqslant c_i \leqslant C$ for some fixed C, then there is such an A, but if $\sup c_i = \infty$ then an appropriate A need not exist.

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2. Notation

In order to express our results succinctly, we introduce a little notation.

Given a sequence $A = (a_i)_{i=1}^{\infty}$ of natural numbers, its *difference sequence* is $D = D(A) = (d_i)_{i=1}^{\infty}$, where $d_i = a_{i+1} - a_i$. We write $\mathscr{D}_{(d,d')}$ for the set of sequences A of natural numbers such that $D(A) = (d_i)_{i=1}^{\infty}$ satisfies $d \leqslant d_i \leqslant d'$ for every i. The set $\mathscr{D}_{(d,d')}$ is precisely the set of *infinite quasi-progressions* with diameter at most $d'-d$ and minimal difference at least d. The existence of *finite* quasi-progressions was studied by Brown et al. [1].

We are also interested in sequences of certain lacunarity: given positive reals (r_i) and integers (c_i) we write $\mathscr{L}_{(r_i,c_i)}$ for the set of sequences $B = (b_i)_{i=1}^{\infty}$ with $b_{i+1} \geqslant r_i b_i - c_i$ for every i. Furthermore, let us write $\mathscr{L}_{(r_i)}$ for $\mathscr{L}_{(r_i,0)}$ and $\mathscr{L}_{(r)}$ for $\mathscr{L}_{(r_i,0)}$ with $r_i = r$ for every i.

3. Meeting the sums $A + A$

To start with, we prove a strengthening of the observation of Erdős and Graham [2].

Theorem 1. *If* $\sup c_i < \infty$ *and* $B \in \mathscr{L}_{(2,c_i)}$ *then* $(A+A)\cap B = \emptyset$ *for some* $A \in \mathscr{D}_{(2,3)}$.

Proof. We may assume that $c_1 = c_2 = \cdots = c$ for some natural number c. If i is large enough then $b_i \geqslant c+1$ and $b_{i+1} > b_i + c$. Let i_0 be the smallest value of $i \geqslant 1$ such that $b_i \geqslant c+1$ and $b_j < b_{j+1} - c$ for $j \geqslant i$. Let us define a sequence $A = (a_1, a_2, \cdots)$ as follows. Set $a_1 = b_{i_0} + 1$. Suppose we have defined $a_1, \ldots, a_{k-1}$, where $k \geqslant 2$. Let $i = i(k)$ be given by $b_i - c - 3 \leqslant a_{k-1} \leqslant b_{i+1} - c - 4$ and set

$$a_k = \begin{cases} a_{k-1} + 3 & \text{if } b_{i+1} = a_{k-1} + 2 + a_j \text{ for some } j \leqslant k, \\ a_{k-1} + 2 & \text{otherwise.} \end{cases}$$

It is clear that $A \in \mathscr{D}_{(2,3)}$, so to complete our proof it suffices to prove that $b_i \notin A+A$ for all i.

Suppose that $b_{i+1} = a_j + a_k$ for some i, j and k, with $j \leqslant k$. Then, by the definition of A, we have $i \geqslant i_0$ and $a_k \geqslant b_{i+1}/2 \geqslant b_i - c/2 \geqslant b_i - c$. Since, $a_j \geqslant a_1 = b_{i_0} + 1 \geqslant c+2$, we have $a_k = b_{i+1} - a_j \leqslant b_{i+1} - c - 2$. Therefore,

$$b_i - c - 3 \leqslant a_k - 3 \leqslant a_{k-1} \leqslant a_k - 2 \leqslant b_{i+1} - c - 4.$$

Hence, by the definition of a_k, if $b_{i+1} = a_{k-1} + 2 + a_h$ for some h, $1 \leqslant h \leqslant k$ then $a_k = a_{k-1} + 3$. But then $b_{i+1} = a_j + a_k = a_j + a_{k-1} + 3$, so $a_h = a_j + 1$ which is impossible since $A \in \mathscr{D}_{(2,3)}$. If, on the other hand, $b_{i+1} \neq a_{k-1} + 2 + a_h$ for all h, $1 \leqslant h \leqslant k$, then $a_k = a_{k-1} + 2$ so $b_{i+1} = a_j + a_k = a_j + a_{k-1} + 2$, which is a contradiction since $j \leqslant k$. $\square$

The first term of the sequence A constructed in the proof above is rather large; it is not clear how small an a_1 we could take so that $(A+A) \cap B = \emptyset$ for some $A = (a_i)_{i=1}^{\infty} \in \mathscr{D}_{(2,3)}$.

Our next aim is to prove the somewhat surprising result that the theorem above is best possible.

Theorem 2. *Let $(c_i)_{i=1}^{\infty}$ be a sequence of integers with $\sup c_i = \infty$ and let b be a natural number at least 2. Then there is a sequence $B = (b_i)_{i=1}^{\infty} \in \mathscr{L}_{(2,c_i)}$ with $b_1 = b$ such that if $A \in \mathscr{D}_{(2,3)}$ then $(A+A) \cap B \neq \emptyset$.*

Proof. Let $1 \leqslant c_{n_1} < c_{n_2} < \cdots$ be an increasing subsequence of $(c_i)_{i=1}^{\infty}$. Put $t_0 = 0$ and for $j \geqslant 1$ set $t_j = \sum_1^j w_{n_i}$. Let us define a sequence $B = (b_1, b_2, \ldots)$ as follows. Having defined $b_1 = b < b_2 < \cdots < b_{j-1}$ for some $j \geqslant 2$, set

$$b_j = \begin{cases} 2b_{j-1} - i + t_{h-1} & \text{if } j = n_i \text{ and } t_{h-1} < i \leqslant t_h - 2, \\ 3b_{j-1} - 2 & \text{if } j = n_{t_h - 1}, \\ 3b_{j-1} & \text{if } j = n_{t_h}, \\ 3b_{j-1} + 3|w_{j-1}| & \text{otherwise.} \end{cases}$$

It is easy to check that $B \in \mathscr{L}_{(2,c_i)}$.

Suppose that, contrary to the assertion, there is a sequence $A = (a_1, a_2, \ldots) \in \mathscr{D}_{(2,3)}$ with $(A+A) \cap B = \emptyset$. Since

$$b_{n_{t_h-1}} \equiv 1 \,(\mathrm{mod}\, 3), \qquad b_{n_{t_h}} \equiv 0 \,(\mathrm{mod}\, 3) \quad \text{and} \quad b_{n_{t_h}+1} \equiv 2 \,(\mathrm{mod}\, 3),$$

the assertion holds if $D(A) = (3, 3, 3, \ldots)$. Hence, we may assume that $D(A) \neq (3, 3, 3, \ldots)$, i.e. $a_{s+1} - a_s = 2$ for some $s \geqslant 1$.

Let h be an index such that $n_h \geqslant 2a_{s+1}$ and $b_{j-1} > a_1 + a_s + 1$, where $j = n_{t_{h-1}} + 2a_s + 2 \leqslant n_{t_h} - 2$. Then $c_{n_h} \geqslant 2a_{s+1}$ as $\{c_{n_i}\}$ is increasing. Since $b_j = 2b_{j-1} - 2a_s - 2 = 2(b_{j-1} - a_s - 1)$, we have $b_{j-1} - a_s - 1 \notin A$. Also, since both a_s and a_{s+1} are in A, we have that both $b_{j-1} - a_s$ and $b_{j-1} - a_{s+1}$ are not in A, otherwise $b_{j-1} \in A + A$. Therefore, the three consecutive integers $b_{j-1} - a_s - 2$, $b_{j-1} - a_s - 1$ and $b_{j-1} - a_s$ are not in A, although $b_{j-1} - a_s - 2 \geqslant a_1$, contradicting that $A \in \mathscr{D}_{(2,3)}$. $\square$

Theorems 1 and 2 show that $(2, c_i)_{i=1}^{\infty}$ is such that for all $B \in \mathscr{L}_{(2,c_i)}$ there is a sequence $A \in \mathscr{D}_{(2,3)}$ with $(A+A) \cap B = \emptyset$ if and only if $\sup c_i < \infty$.

4. Meeting the sums $A + A + A$

Let us turn to the question of Erdős and Graham [2] mentioned in the Introduction. As our final result shows, no lacunarity condition on B implies that $(A+A+A) \cap B = \emptyset$ for some $A \in \mathscr{D}_{(2,3)}$.

Theorem 3. *Let $1 \leqslant r_1 < r_2 < \cdots$ be a sequence of integers. Then, there is a sequence b_i with $B \in \mathscr{L}_{(r_i)}$ such that if $A \in \mathscr{D}_{(2,3)}$ then $(A+A+A) \cap B \neq \emptyset$.*

Proof. Let us define a sequence $B = (b_i)_{i=1}^{\infty}$ by setting $b_1 = 1$ and $b_i = 6ir_ib_{i-1} + i - 3\lfloor i/3 \rfloor$ for $i \geqslant 2$. We claim that is sequence B will do.

Suppose, for a contradiction, that $(A+A+A) \cap B = \emptyset$ for some $A \in \mathscr{D}_{(2,3)}$. Then, $A+A$ does not contain three consecutive integers, since otherwise $A+A+A$ contains all sufficiently large integers. Furthermore, $A \neq \{a_1, a_1+2, a_1+4, \ldots\}$ since otherwise $A+A+A = \{3a_1, 3a_1+2, 3a_1+4, \ldots\}$. Also, $A \neq \{a_1, a_1+3, a_1+6, \ldots\}$ since otherwise $A+A+A = \{3a_1, 3a_1+2, 3a_1+6, \ldots\}$.

This shows that both 2 and 3 occur infinitely often in the difference sequence $D = D(A) = (d_i)_{i=1}^{\infty}$. Let us check next that $d_i = d_{i+1} = 2$ cannot hold. Indeed, otherwise $A+A$ contains the three consecutive integers $a_{i+1} + a_j = a_i + a_j + 2$, $a_i + b_{j+1} = a_i + a_j + 3$ and $a_{i+2} + a_j = a_i + a_j + 4$, where $a_{j+1} = a_j + 3$.

Assuming, as we may, that $a_2 = a_1 + 2$, i.e. $d_1 = 2$, the sequence D is of the form 2, 3, 3, …, 3, 2, 3, …, 3, 2, 3, …, i.e. 2 is followed by a block of 3 of length l_1, then 2 again, followed by a block of 3 of length l_2, and so on.

Suppose first that $l_1 = l_2 = \cdots = m$. Then

$$A+A+A \supset \{3a_2 + (3l+2)n + 3(m_1+m_2+m_3) :$$
$$n = 0, 1, \ldots, 0 \leqslant m_i \leqslant m, \ i = 1, 2, 3\};$$

in particular, $A+A+A$ contains all sufficiently large integers except those congruent to $3(a_1+1)$ modulo $3m+2$. From here it is but a short step to a contradiction. Indeed, let j be sufficiently large. With $i = 3j(3m+2)$, we have $i - 3\lfloor i/3 \rfloor = 0$ and $b_i \equiv 0 \pmod{3m+2}$, so $b_i \notin A+A+A$ implies $3(a_1+1) \equiv 0 \pmod{3m+2}$. Furthermore, with $h = (3j+1)(3m+2)$ we have $h - 3\lfloor h/3 \rfloor = 2$ implies the contradiction that $3(a_1+1) \equiv 2 \pmod{3m+2}$.

Since not all l_i are equal, we may assume that $l_i \leqslant l_j - 1$ for some i and j, so

$$A \supset \{a, a+2, a+5, a+8, \ldots, a+3l_i+2, a+3l_i+4\}$$
$$\cup \{b, b+3, b+6, \ldots, b+3l_j\}$$

for some natural numbers a and b. But then $A+A$ contains the three consecutive integers $(a+3l_i+2)+b$, $a+(b+3(l_i+1))$ and $(a+3l_i+4)+b$. Therefore, $A+A+A$ contains all sufficiently large integers, contradicting that $(A+A+A) \cap B = \emptyset$. $\square$

5. Final remarks

The problems discussed above are special cases of a much more general problem.

Given natural numbers d and d' with $d < d'$, is there an $r > 0$ such that if $B \in \mathscr{L}_{(r)}$ then there is a sequence $A \in \mathscr{D}_{(d,d')}$ with $(A+A) \cap B = \emptyset$? If there is such an r then what is $r(d, d') = r_2(d, d')$, the smallest value of r that we can take?

We have just proved that $r(2, 3) = 2$. It is easy to see that if $r(d, d+1)$ exists for $d \geqslant 3$, then $r(d, d+1) \leqslant 2$. However, we cannot decide whether $r(d, d+1)$ exists for every $d \geqslant 3$, and we do not know for what values of d we have $r(d, d+1) < 2$.

Also, what can we say about the corresponding function $r_k(d, d')$ concerning sums $A + \cdots + A$ of k terms rather than sums $A + A$? As shown above, $r_3(2, 3)$ does not exist.

References

[1] T.C. Brown, P. Erdős and A.R. Freedman, Quasi-progressions and descending waves, J. Combin. Theory Ser. A 53 (1990) 81–95.

[2] P. Erdős and R.L. Graham, Old and new problems and results in combinatorial number theory, Monographie No. 28, de L'Enseignement Mathématique, 1980.